Linear eqn with variable coefficients

Section-1

Introduction:

A linear differential eqn of order n with variable coefficients is an eqn of the form.

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \dots + a_n(x)y = b(x)$$

where $a_0, a_1 \dots a_n, b$ are complex valued functions on some real interval I. Points where $a_0(x)=0$ are called singular points and often the eqn requires special consideration at such points. Therefore in this case assume that $a_0(x) \neq 0$ on I. By dividing by a_0 we can obtain an eqn of the same form, but with a_0 replaced by the constant 1. Thus we consider the eqn.

$$y^{(n)} + a_1(x)y^{(n-1)} + \dots + a_n(x)y = b(x) \rightarrow (1)$$

As in the case when $a_1 \dots a_n$ are constants we designate the left side of (1) by $L(y)$. Thus

$$L(y) = y^{(n)} + a_1(x)y^{(n-1)} + \dots + a_n(x)y \rightarrow (2)$$

And (1) becomes simply $L(y) = b(x)$. If $b(x)=0$ $\forall$ x in I we say $L(y)=0$ is a homogeneous eqn. whereas if $b(x) \neq 0$ for some x in I. The eqn $L(y)=b(x)$ is called a non-homogeneous eqn.

We give a meaning to L itself as an operator which takes each function ϕ, which has 'n' derivatives on I, into the function $L(\phi)$ on I whose value at x is given by

$$L(\phi)(x) = \phi^{(n)}(x) + a_1(x)\phi^{(n-1)}(x) + \dots + a_n(x)\phi(x)$$

Thus a soln of ① on I is a function φ on I which has n derivatives there and which satisfies $L(\varphi)=b$.

In this chapter we assume that the complex valued functions $a_1, a_2 \dots a_n, b$ are continuous on some real interval I and $L(y)$ will always denote the expression of ②

Section-2

Initial value problem for the homogeneous eqn

Theorem: 1 Existence Theorem.

Let $a_1, a_2, \dots a_n$ be continuous function on an interval I containing the point x_0. If $\alpha_1 \dots \alpha_n$ are any n constants there exists a soln φ of

$$L(y) = y^{(n)} + a_1(x)\, y^{(n-1)} + \dots + a_n(x)\, y = 0$$

on I satisfying

$$\varphi(x_0) = \alpha_1, \ \varphi'(x_0) = \alpha_2 \dots \varphi^{(n-1)}(x_0) = \alpha_n$$

Proof:

Let $\varphi_1, \varphi_2 \dots \varphi_n$ be any set of n linearly independent soln of

$$L(y) = y^{(n)} + a_1(x) y^{(n-1)} + \dots + a_n(x) y = 0$$

It will be S.T $\exists$ unique constants $c_1, c_2 \dots c_n$ such that

$$\varphi = c_1\varphi_1 + c_2\varphi_2 + \dots + c_n\varphi_n \text{ is a soln of } L(y)=0$$

satisfying the condition given by ①

These constants would have to satisfy.

$$c_1\varphi_1(x_0) + c_2\varphi_2(x_0) + \dots + c_n\varphi_n(x_0) = \alpha_1$$

$$c_1\varphi_1'(x_0) + c_2\varphi_2'(x_0) + \dots + c_n\varphi_n'(x_0) = \alpha_2$$

$$\vdots$$

$$c_1\varphi_1^{(n-1)}(x_0) + c_2\varphi_2^{(n-1)}(x_0) + \dots + c_n\varphi_n^{(n-1)}(x_0) = \alpha_n$$

This is a system of n linear equations for $c_1, c_2 \ldots c_n$. The displacement of the coefficient is just $W(\varphi_1, \varphi_2 \ldots \varphi_n)(x_0)$ which is not zero.

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$\therefore$ There is a unique set of constants $c_1, c_2 \ldots c_n$ satisfying

for this choice $c_1, c_2 \ldots c_n$ the function

$$\varphi = c_1\varphi_1 + c_2\varphi_2 + \cdots + c_n\varphi_n$$

Gives the desired soln.

Note:

There are two important things about the thm

(i) The soln exists on the entire interval I. where $a_1, a_2 \ldots a_n$ are continuous.

(ii) Every initial value problem has a soln. Neither of these result may be true if the coefficient of $y^{(n)}$ vanishes somewhere in I.

Example:

Consider the eqn $xy' + y = 0$ $\Rightarrow \frac{dy}{dx} = -y/x$

$\frac{dy}{dx} + \frac{1}{x}y = 0$ $\int \frac{dx}{x}$ $IF = e^{\ln x} = e^{\ldots} = x$

where coefficients are continuous for all real x. This eqn and the initial condition $y(1) = 1$.

The soln φ_1

$y \cdot x = 0 + c$

$y = c/x$

$$\varphi_1(x) = \frac{1}{x}$$

But this soln exists only for $0 < x < \infty$. Also if φ is any soln then,

$$x\varphi(x) = c, \quad \text{where } c \text{ is constant.}$$

Thus only the trivial soln $(c=0)$ exists at the origin

$\Rightarrow$ The only IVP

$$xy' + y = 0, \quad y(0) = \alpha,$$

which has a soln is the one for which $\alpha = 0$.

Note:

$$\|\varphi(x)\|^2 = |\varphi(x)|^2 + |\varphi'(x)|^2 + \dots + |\varphi^{(n-1)}(x)|^2$$

Theorem: 2

Let $b_1, b_2 \dots b_n$ non negative constants such that for x in I $|a_j(x)| \le b_j$ $(j=1,2\dots n)$ and define k by $k = 1 + b_1 + b_2 + \dots + b_n$. If x_0 is a point in I and φ is a soln of $L(y)=0$ on I. Then

$$\|\varphi(x_0)\| e^{-k|x-x_0|} \le \|\varphi(x)\| \le \|\varphi(x_0)\| e^{k|x-x_0|} \quad \text{for all } x \text{ in } I$$

Proof:

Since $L(\varphi)=0$

We have,

$$L(\varphi) = \varphi^{(n)}(x) + a_1(x)\varphi^{(n-1)}(x) + \dots + a_n(x)\varphi(x) = 0$$

$$\therefore \varphi^{(n)}(x) = -a_1(x)\varphi^{(n-1)}(x) + \dots + a_n(x)\varphi(x)$$

And Therefore,

$$|\varphi^{(n)}(x)| = |-a_1(x)\varphi^{(n-1)}(x) + \dots + a_n(x)\varphi(x)|$$

$$\le |a_1(x)\varphi^{(n-1)}(x)| + \dots + |a_n(x)\varphi(x)|$$

$$\le |a_1(x)||\varphi^{(n-1)}(x)| + \dots + |a_n(x)||\varphi(x)|$$

$$|\varphi^{(n)}(x)| \le b_1|\varphi^{(n-1)}(x)| + \dots + b_n|\varphi(x)| \rightarrow (1)$$

Let $u(x) = \|\varphi(x)\|^2$

$$u(x) = |\varphi(x)|^2 + |\varphi'(x)|^2 + \dots + |\varphi^{(n-1)}(x)|^2$$

$$\Rightarrow u = \varphi\bar{\varphi} + \varphi'\bar{\varphi}' + \dots + \varphi^{(n-1)}\overline{\varphi^{(n-1)}}$$

Diff w.r.t x we get,

$$u' = \varphi'\bar{\varphi} + \varphi\cdot\bar{\varphi}' + \varphi''\bar{\varphi}' + \varphi'\bar{\varphi}'' + \dots + \varphi^{(n)}\overline{\varphi^{(n-1)}} + \varphi^{(n-1)}\overline{\varphi^{(n)}}$$

$$|u'| \le |\varphi'||\varphi| + |\varphi||\varphi'| + |\varphi''||\varphi'| + |\varphi'||\varphi''| + \dots + |\varphi^{(n)}||\varphi^{(n-1)}| + |\varphi^{(n-1)}||\varphi^{(n)}|$$

$$|u'| \le 2|\varphi'||\varphi| + 2|\varphi'||\varphi''| + \dots + 2|\varphi^{(n-1)}||\varphi^{(n)}| \rightarrow (2)$$

Sub ① in ②

$$|u'| \le 2|\varphi||\varphi'| + 2|\varphi'||\varphi''| + \dots + 2|\varphi^{(n-1)}|\left[b_1|\varphi^{(n-1)}(x)| + \dots + b_n|\varphi(x)|\right]$$

$$|u'| \le 2|\varphi||\varphi'| + 2|\varphi'||\varphi''| + \dots + 2b_1|\varphi^{(n-1)}|^2 + 2b_2|\varphi^{(n-1)}||\varphi^{(n-2)}| + \dots + 2b_n|\varphi^{(n-1)}||\varphi|$$

using the inequality

$$2|b||c| \le |b|^2 + |c|^2$$

we get,

$$|u'| \le [|\varphi|^2 + |\varphi'|^2] + [|\varphi'|^2 + |\varphi''|^2] + \dots + 2b_1|\varphi^{(n-1)}|^2 + 2b_1|\varphi^{(n-1)}||\varphi^{(n-2)}| + \dots + 2b_n|\varphi^{(n-1)}||\varphi|$$

$$\le [|\varphi|^2 + |\varphi'|^2] + [|\varphi'|^2 + |\varphi''|^2] + \dots + 2b_1|\varphi^{(n-1)}|^2 + b_2[|\varphi^{(n-1)}|^2 + |\varphi^{(n-2)}|^2] + \dots + b_n[|\varphi^{(n-1)}|^2 + |\varphi|^2]$$

$$= (1+b_n)|\varphi|^2 + (2+b_{n-1})|\varphi'|^2 + \dots + (2+b_2)|\varphi^{(n-2)}|^2 + \dots + (1+2b_1+b_2+\dots+b_n)|\varphi^{(n-1)}|^2$$

$$|u'| = 2(1+b_1+b_2+\dots+b_n)|\varphi|^2 + 2(1+b_1+b_2+\dots+b_n)|\varphi'|^2 + \dots + 2(1+b_1+b_2+\dots+b_n)|\varphi^{(n-1)}|^2$$

$$= 2k|\varphi|^2 + 2k|\varphi'|^2 + \dots + 2k|\varphi^{(n-1)}|^2$$

$$= 2k\left[|\varphi|^2 + |\varphi'|^2 + \dots + |\varphi^{(n-1)}|^2\right]$$

$$|u'| \le 2ku$$

$$\Rightarrow -2ku \le u' \le 2ku$$

consider the right inequality which can be written as

$$u' - 2ku \le 0$$

$$\Rightarrow e^{-2ku}(u' - 2ku) \le 0$$

$$\Rightarrow (e^{-2ku}u)' \le 0$$

If $x > x_0$, we integrate from x_0 to x and get,

$$[e^{-2ku}u(x)]_{x_0}^{x} \le 0$$

$$e^{-2kx}u(x) - e^{-2kx_0}u(x_0) \le 0$$

$$e^{-2Kx} u(x) \le e^{-2Kx_0} u(x_0)$$

$$u(x) \le e^{-2Kx_0 + 2Kx} u(x_0)$$

$$u(x) \le u(x_0) e^{2K(x-x_0)}$$

$$\Rightarrow \|\varphi(x)\|^2 \le \|\varphi(x_0)\|^2 e^{2K(x-x_0)}$$

$$\Rightarrow \|\varphi(x)\|^2 \le \|\varphi(x_0)\|^2 e^{K|x-x_0|} \rightarrow (3) \text{ if } x > x_0$$

Similarly the left inequality

$$\|\varphi(x_0)\| e^{K|x-x_0|} \le \|\varphi(x)\| \rightarrow (4) \text{ if } x < x_0$$

Combining (3) and (4) we get,

$$\|\varphi(x_0)\| e^{-K|x-x_0|} \le \|\varphi(x)\| \le \|\varphi(x_0)\| e^{K|x-x_0|}$$

Hence the Theorem.

Remark:

If I is a closed bounded interval that is of the form $a \le x \le b$ with a, b real and if the a_j are continuous then there always exist finite constants b_j, such that,

$$|a_j(x)| \le b_j \quad \text{on } I$$

Theorem: 3 Uniqueness Theorem.

Let x_0 be in I and let $\alpha_1, \alpha_2 \ldots \alpha_n$ be any n consta[nts]. There is atmost on soln φ of $L(y)=0$ on I satisfying

$$\varphi(x_0) = \alpha_1, \quad \varphi'(x_0) = \alpha_2 \quad \ldots \quad \varphi^{(n-1)}(x_0) = \alpha_n$$

Proof:

Suppose, φ, ψ are two soln of $L(y)=0$ on I satisfyin[g] the conditions at x_0.

Then $\theta = \varphi - \psi$, satisfy $L[\theta] = 0$

and $\theta(x_0) = \theta'(x_0) \ldots = \theta^{(n-1)}(x_0) = 0$

Thus $\|\theta(x_0)\| = 0$

From the inequality.

$$\|\theta(x_0)\| e^{-k|x-x_0|} \le \|\varphi(x)\| \le \|\theta(x_0)\| e^{k|x-x_0|}$$

We get,

$$\|\theta(x)\| = 0, \quad \forall\ x \text{ in } I$$

ie) $\theta(x) = 0, \quad \forall\ x$ in I

$\Rightarrow \varphi = \psi$

Note:

$a_1(x) = x$ is not bounded on $0 \le x < \infty$ and $a_1(x) = 1/x$ is not bounded on $0 < x \le 1$

Section-3

Solns of the homogeneous equation.

If $\varphi_1, \varphi_2 \dots \varphi_m$ are any m solns of the n^{th} order eqn $L(y) = 0$ on an interval I and $c_1, c_2 \dots c_m$ are any m constants Then.

$$L(c_1\varphi_1 + c_2\varphi_2 + \dots + c_m\varphi_m) = c_1 L(\varphi_1) + \dots + c_m L(\varphi_m)$$

$\Rightarrow c_1\varphi_1 + c_2\varphi_2 + \dots + c_m\varphi_m$ is also a soln.

Any linear combination of solns is again a soln. The trivial soln is the function which is identically zero on I.

Theorem: 4

There exist n linearly independent solns of $L(y) = 0$ on I

Proof:

Let x_0 be a point in I.

by Thm 1 There is a soln φ_1 of $L(y) = 0$ satisfying

$$\varphi_1(x_0) = 1, \quad \varphi_1'(x_0) = 0 \quad \dots \quad \varphi^{(n-1)}(x_0) = 0$$

In general for each $i = 1, 2 \dots n$ There is a soln φ_i satisfying

$$\varphi_i^{(i-1)}(x_0) = 1, \quad \varphi_i^{(j-1)}(x_0) = 0 \quad j \ne i \quad \to (1)$$

The Solns $\varphi_1, \varphi_2 \dots \varphi_n$ are linearly independent on J.

Suppose There are constants $c_1, c_2 \dots c_n$ such That,

$$c_1\varphi_1(x) + c_2\varphi_2(x) + \dots + c_n\varphi_n(x) = 0 \rightarrow ②$$

Diff w.r.t x weget,

$$\left.\begin{aligned} c_1\varphi_1'(x) + c_2\varphi_2'(x) + \dots + c_n\varphi_n'(x) &= 0 \\ c_1\varphi_1''(x) + c_2\varphi_2''(x) + \dots + c_n\varphi_n''(x) &= 0 \\ &\vdots \\ c_1\varphi_1^{(n-1)}(x) + c_2\varphi_2^{(n-1)}(x) + \dots + c_n\varphi_n^{(n-1)}(x) &= 0 \end{aligned}\right\} \rightarrow ③$$

In Particular eqn ② and ③ must hold at x_0. Putting $x=x_0$ and using ①

weget,

$$c_1(1) + 0 + \dots + 0 = 0$$

$$\therefore c_1 = 0$$

Putting $x = x_0$ in eqn ③

We obtain,

$$c_2 = c_3 = \dots = c_n = 0$$

Thus The Solns $\varphi_1, \varphi_2 \dots \varphi_n$ are linearly independent.

Theorem: 5

Let $\varphi_1, \varphi_2 \dots \varphi_n$ be The n Solns of $L(y)=0$ on I satisfying $\varphi_i^{(i-1)}(x_0) = 1$, $\varphi_i^{(j-1)}(x_0) = 0$ $i \neq j$. If φ is any Soln of $L(y)=0$ on I. There are n constants $c_1, c_2 \dots c_n$ Such That $\varphi = c_1\varphi_1 + c_2\varphi_2 + \dots + c_n\varphi_n$

Proof:

Let $\varphi(x_0) = \alpha_1$, $\varphi'(x_0) = \alpha_2 \dots \varphi^{(n-1)}(x_0) = \alpha_n$

Consider The function,

$$\psi = \alpha_1\varphi_1 + \alpha_2\varphi_2 + \dots + \alpha_n\varphi_n$$

It is a Soln of $L(y) = 0$

$$\therefore \psi(x_0) = \alpha_1\varphi_1(x_0) + \alpha_2\varphi_2(x_0) + \dots + \alpha_n\varphi_n(x_0) = \alpha_1 \rightarrow (*)$$

Since $\varphi_1(x_0) = 1$, $\varphi_2(x_0) = 0 \dots \varphi_n(x_0) = 0$

(*) $\Rightarrow$ $\psi(x_0) = \alpha_1$, $\psi'(x_0) = \alpha_2 \dots \psi^{(n-1)}(x_0) = 0$

Thus ψ is a Soln of $L(y) = 0$. Satisfying The Same initial conditions at x_0 as φ.

By uniqueness Thm, we must have,

$$\varphi = \psi \quad \text{ie)} \quad \varphi = \alpha_1\varphi_1 + \alpha_2\varphi_2 + \dots + \alpha_n\varphi_n$$

We have proved The Thm with The constants

$$c_1 = \alpha_1, \; c_2 = \alpha_2 \dots \; c_n = \alpha_n$$

Remark:

A Set of function which has The property That if φ_1, φ_2 belongs to The Set and c_1, c_2 are any two constant Then,

$c_1\varphi_1 + c_2\varphi_2$ belongs to the Set also is Called a linear space of functions.

We have seen That The Set of all soln of $L(y) = 0$ on an interval I is a linear of soln.

If a linear space of fun contains n functions $\varphi_1, \varphi_2 \dots \varphi_n$ which are linearly independent and Such That every function in The Space can be represented as a linear Combinations of These Then,

$\varphi_1, \varphi_2 \dots \varphi_n$ is Called a basis for The linear Space and The dimension of The linear Space The integer n.

Thm (5) tells That The functions $\varphi_1, \varphi_2 \dots \varphi_n$ satisfying The initial condition

$$\varphi_i^{(i-1)} = 1, \quad \varphi_i^{(j-1)} = 0 \qquad j \neq i$$

from a basis for The soln of $L(y) = 0$ on I and This linear Space of fun has dimension n

1.a) Consider the eqn $y'' + \frac{1}{x} y' - \frac{1}{x^2} y = 0$ for $x > 0$

(i) S.T there is a soln of the form x^r where r is a constant.

(ii) Find two L.I solns for $x > 0$. P.T they are L.I.

(iii) Find the two solns ϕ_1, ϕ_2 satisfying

$\phi_1(1) = 1, \quad \phi_2(1) = 0$

$\phi_1'(1) = 0, \quad \phi_2'(1) = 1$

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Soln:

Given eqn is $y'' + \frac{1}{x} y' - \frac{1}{x^2} y = 0$

$$x^2 y'' + x y' - y = 0 \rightarrow (1)$$

Let $\phi(x) = x^r$ be a soln of this eqn.

$\Rightarrow \phi'(x) = r x^{r-1}$

$\phi''(x) = r(r-1) x^{r-2}$

$(1) \Rightarrow \quad x^2 \phi'' + x \phi' - \phi = 0 \qquad (\because \phi \text{ is soln})$

$$x^2 r(r-1) x^{r-2} + x x^{r-1} r - x^r = 0$$

$$x^r r(r-1) + x^r r - x^r = 0$$

$$x^r (r^2 - r + r - 1) = 0$$

$$r^2 - 1 = 0 \qquad x^r \neq 0$$

$$r = \pm 1$$

$\therefore \phi_1(x) = x$ and $\phi_2(x) = x^{-1} = \frac{1}{x}$ are two soln.

$$W(\phi_1, \phi_2) = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} = \begin{vmatrix} x & 1/x \\ 1 & -1/x^2 \end{vmatrix}$$

$$= x\left(\frac{-1}{x^2}\right) - \frac{1}{x} = -\frac{2}{x} \neq 0$$

i.e) $W(\phi_1, \phi_2) \neq 0$

$\therefore$ The two solns are linearly independent.

$\therefore \phi_1(x) = c_1 x + c_2 \frac{1}{x}$, c_1, c_2 any constants is a soln of the eqn

Given $\phi_1(1) = 1$

$\phi_1(1) = c_1 + c_2$

$$c_1 + c_2 = 1 \rightarrow (2)$$

$\varphi_1'(x) = c_1 - c_2 \frac{1}{x^2}$ Given $\varphi_1'(1) = 0$

$\varphi_1'(1) = c_1 - c_2$

$c_1 - c_2 = 0 \rightarrow (3)$

$(2) \Rightarrow c_1 + c_2 = 1$

$(3) \Rightarrow c_1 - c_2 = 0$

$2c_1 = 1$

$c_1 = 1/2$

$\Rightarrow c_2 = 1/2$

$\therefore \varphi(x) = \frac{1}{2}x + \frac{1}{2}\cdot\frac{1}{x} \Rightarrow \frac{1}{2}(x + \frac{1}{x})$

HW $(3x-1)^2 y'' + (9x-3)y' - 9y = 0$

replace x by $3x-1$ in (1) (2)

Given $\varphi_2(1) = 0$

$\varphi_2(x) = c_1 x + c_2 \frac{1}{x}$

$\varphi_2(1) = c_1 + c_2$

$c_1 + c_2 = 0 \rightarrow (4)$

$\varphi_2'(x) = c_1 - \frac{c_2}{x^2}$ Given $\varphi_2'(1) = 1$

$\varphi_2'(1) = c_1 - c_2$

$c_1 - c_2 = 1 \rightarrow (5)$

$(4) \Rightarrow c_1 + c_2 = 0$

$(5) \Rightarrow c_1 - c_2 = 1$

$2c_1 = 1$

$c_1 = \frac{1}{2}$

$\Rightarrow c_2 = -\frac{1}{2}$

$\therefore \varphi(x) = \frac{1}{2}x - \frac{1}{2}\cdot\frac{1}{x} \Rightarrow \frac{1}{2}(x - 1/x)$

..b) The eqn $y' + a(x)y = 0$ has for a soln $\varphi(x) = \exp[-\int_{x_0}^{x} a(t)\,dt]$, Here $a(t)$ be continuous on an interval I containing x_0. This suggest trying to find a soln of $L(y) = y'' + a_1(x)y' + a_2(x)y = 0$ of the form, $\varphi(x) = \exp[\int_{x_0}^{x} p(t)\,dt]$. where p is a function, to determined. S.T φ is a soln of $L(y) = 0$ iff p satisfies the first order non-linear eqn $y' = -y^2 - a_1(x)y - a_2(x)$,

This eqn is called Riccati eqn

Soln:

Given $\varphi(x) = \exp\left[\int_{x_0}^{x} p(t)\,dt\right]$

$$\varphi'(x) = \exp\left[\int_{x_0}^{x} p(t)\,dt\right] p(x)$$

$$\varphi'(x) = \varphi(x).p(x)$$

ie) $\varphi' = \varphi p$

$$\varphi'' = \varphi' p + p'\varphi = \varphi p \cdot p + \varphi' p$$

$$\varphi'' = \varphi p^2 + p'\varphi$$

φ is a soln of $L(y)=0$ iff $\varphi'' + a_1(x)\varphi' + a_2(x)\varphi = 0$

ie) iff $\Leftrightarrow (\varphi p^2 + p'\varphi) + a_1(x)\varphi p + a_2(x)\varphi = 0$

$$\Leftrightarrow \varphi(p^2 + p' + a_1(x)p + a_2(x)) = 0$$

$$\Leftrightarrow p^2 + p' + a_1(x)p + a_2(x) = 0$$

$$\Leftrightarrow p' = -p^2 - a_1(x)p - a_2(x)$$

$\Leftrightarrow$ p satisfies the first order non-linear eqn.

$$y' = -y^2 - a_1(x)y - a_2(x).$$

1.c) Let $\varphi_1, \varphi_2 \dots \varphi_n$ be n continuous function on interval $a \le x \dots$

Let $\alpha_{ij} = \int_a^b \overline{\varphi_i(x)}\,\varphi_j(x)\,dx$ $(i,j = 1,2 \dots n)$ and let Δ denote the

determinant
$$\Delta = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & & & \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{vmatrix}$$

P.T $\varphi_1, \varphi_2 \dots \varphi_n$ are linearly independent on $a \le x \le b$ iff $\Delta \ne 0$

Soln:

Suppose $\Delta \ne 0$ and consider,

$$c_1\varphi_1 + c_2\varphi_2 + \dots + c_n\varphi_n = 0 \rightarrow (1)$$

multiplying (1) by $\bar{\varphi}_1$ throughout and integrating from a to b we get,

$$c_1\int_a^b \bar{\varphi}_1\varphi_1(x)\,dx + c_2\int_a^b \bar{\varphi}_1\varphi_2\,dx + \dots + c_n\int_a^b \bar{\varphi}_1\varphi_n\,dx = 0.$$

ie) $c_1\alpha_{11} + c_2\alpha_{12} + \dots + c_n\alpha_{1n} = 0$

Similarly multiplying ① interm by $\bar{\Phi}_2 \ldots \bar{\varphi}_n$ and integrating from a to b.

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$$c_1\alpha_{21} + c_2\alpha_{22} + \cdots + c_n\alpha_{2n} = 0$$
$$\vdots$$
$$c_1\alpha_{n1} + c_2\alpha_{n2} + \cdots + c_n\alpha_{nn} = 0$$

But The determinant of the coefficients of above system of eqn is just Δ which is not zero.

$\therefore$ The only soln of The System is

$$c_1 = c_2 = \cdots = c_n = 0$$

$\therefore \varphi_1, \varphi_2 \ldots \varphi_n$ are linearly independent.

conversely if,

$\varphi_1, \varphi_2 \ldots \varphi_n$ are linearly independent and if possible $\Delta = 0$

Then There are constants $c_1, c_2 \ldots c_n$ not all zero Such That.

$$c_1\alpha_{11} + c_2\alpha_{12} + \cdots + c_n\alpha_{1n} = 0$$
$$c_1\alpha_{21} + c_2\alpha_{22} + \cdots + c_n\alpha_{2n} = 0$$
$$\vdots$$
$$c_1\alpha_{n1} + c_2\alpha_{n2} + \cdots + c_n\alpha_{nn} = 0$$

multiplying ① by $\bar{c}_1$ The Second by $\bar{c}_2$ and so on and adding all The equations we get.

$$\sum_{j=1}^{n}\sum_{i=1}^{n} \bar{c}_i \alpha_{ij} c_j = 0$$

i.e. $$\int_a^b \left| \sum_{i=1}^{n} c_i\varphi_i(x) \right|^2 dx = 0$$

$$\Rightarrow \sum_{i=1}^{n} c_i\varphi_i(x) = 0$$

i.e. $c_1\varphi_1 + c_2\varphi_2 + \cdots + c_n\varphi_n = 0$

This Show That $\varphi_1, \varphi_2 \ldots \varphi_n$ are linearly dependent.

a Contradiction $\Delta \neq 0$.

1.d) Method of removal of first derivative.

Consider the eqn $L(y) = y'' + a_1(x)y' + a_2(x)y = 0$, where a_1, a_2 are continuous on some interval I and a_1 has a continuous derivative there.

(a) If φ is a soln of $L(y) = 0$. Let $\varphi = u\psi$ and determine a differential eqn for u which will make ψ the soln of an eqn in which the first derivative term is absent.

(b) Solve this differential eqn for u.

(c) S.T ψ will then satisfy the eqn $y'' + \alpha(x)y = 0$

where $\alpha = a_2 - \frac{a_1^2}{4} - \frac{a_1'}{2}$

Soln:

Let $\varphi = u\psi$

Since $L(\varphi) = 0$ we get, $L(u\psi) = 0$

i.e) $L(u\psi)'' + a_1(u\psi)' + a_2 u\psi = 0$

$$u\psi'' + 2u'\psi' + u''\psi + a_1\{u\psi' + u'\psi\} + a_2 u\psi = 0$$

$$u\psi'' + \psi'(2u' + a_1 u) + \psi(u'' + a_1 u' + a_2 u) = 0 \quad \to (1)$$

Since the first derivative term is to absent.

$$\therefore 2u' + a_1 u = 0. \quad \to (2)$$

This is linear in u

A soln of this eqn is

$$u = ce^{-\frac{1}{2}\int_{x_0}^{x} a_1(t)\,dt} \quad \to (3)$$

c being any constant,

(1) $\Rightarrow$ $u\psi'' + \psi(u'' + a_1 u' + a_2 u) = 0$

(3) $\Rightarrow$ $u' = ce^{-\frac{1}{2}\int_{x_0}^{x} a_1(t)dt}\left(-\frac{1}{2}a_1\right)$

$$= -\frac{1}{2}a_1 u$$

$$u'' = -\frac{1}{2}[a_1 u' + a_1' u]$$

$$\therefore u'' + a_1 u' + a_2 u = -\frac{1}{2}[a_1 u' + a_1' u] + a_1 u' + a_2 u$$

$$= \frac{a_1 u'}{2} - \frac{a_1' u}{2} + a_2 u$$

1.d) Method of removal of first derivative.

Consider the eqn $L(y) = y'' + a_1(x)y' + a_2(x)y = 0$, where a_1, a_2 are continuous on some interval I and a_1 has a continuous derivative there.

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Let $\phi = u\psi$

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$u\psi'' + \psi'(2u' + a_1 u) + \psi(u'' + a_1 u' + a_2 u) = 0 \quad \to (1)$

Since the first derivative term is to absent.

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This is linear in u

A soln of this eqn is

$u = c\, e^{-\frac{1}{2}\int_{x_0}^{x} a_1(t)dt} \quad \to (3)$

c being any constant,

(1) $\Rightarrow$ $u\psi'' + \psi(u'' + a_1 u' + a_2 u) = 0$

(3) $\Rightarrow$ $u' = c\, e^{-\frac{1}{2}\int_{x_0}^{x} a_1(t)dt} \left(-\frac{1}{2}a_1\right)$

$= -\frac{1}{2}a_1 u$

$u'' = -\frac{1}{2}[a_1 u' + a_1' u]$

$\therefore u'' + a_1 u' + a_2 u = -\frac{1}{2}[a_1 u' + a_1' u] + a_1 u' + a_2 u$

$= \frac{a_1 u'}{2} - \frac{a_1' u}{2} + a_2 u$

$$= \frac{a_1}{2}\left(-\frac{1}{2}a_1 u\right) - \frac{a_1'}{2}u + a_2 u$$

$$= \left(-\frac{a_1^2}{4} - \frac{a_1'}{2} + a_2\right) u$$

Sub in ① we get,

$$u\psi'' + u\psi\left[a_2 - \frac{a_1'}{2} - \frac{a_1^2}{4}\right] = 0$$

ie) $\psi'' + \alpha\psi = 0$, where $\alpha = a_2 - \frac{a_1^2}{2} - \frac{a_1'}{2}$

Section-4

The Wronskian and linear independence.

In order to show that any set of n linearly independent solns of $L(y)=0$ can serve as a basis for the solns of $L(y)=0$. We consider the wronskian $W(\varphi_1, \varphi_2 \dots \varphi_n)$ of any n solns $\varphi_1, \varphi_2 \dots \varphi_n$. The determinant is defined as,

$$W(\varphi_1, \varphi_2 \dots \varphi_n) = \begin{vmatrix} \varphi_1 & \varphi_2 & \dots & \varphi_n \\ \varphi_1' & \varphi_2' & \dots & \varphi_n' \\ \vdots & & & \\ \varphi_1^{(n-1)} & \varphi_2^{(n-1)} & \dots & \varphi_n^{(n-1)} \end{vmatrix}$$

Theorem: 6

If $\varphi_1, \varphi_2 \dots \varphi_n$ are n Solns of $L(y)=0$ on an interval I on an interval I they are linearly independent iff $W(\varphi_1, \varphi_2 \dots \varphi_n)(x) \neq 0 \ \forall\ x$ in I

Proof:

First suppose $W(\varphi_1, \varphi_2 \dots \varphi_n)(x) \neq 0 \ \forall\ x$ in I.

If there are constants $c_1, c_2 \dots c_n$ such that,

$$c_1\varphi_1(x) + c_2\varphi_2(x) + \dots + c_n\varphi_n(x) = 0 \rightarrow ①$$

Then clearly,

$$\left.\begin{aligned} &c_1\varphi_1'(x) + c_2\varphi_2'(x) + \dots + c_n\varphi_n'(x) = 0 \\ &\vdots \\ &c_1\varphi_1^{(n-1)}(x) + c_2\varphi_2^{(n-1)}(x) + \dots + c_n\varphi_n^{(n-1)}(x) = 0 \end{aligned}\right\} \rightarrow ②$$

for fixed x in I the eqn ① and ② are n linear homogeneous equations satisfied by $c_1, c_2 \dots c_n$. The determinant of the coefficient is $W(\varphi_1, \varphi_2 \dots \varphi_n)(x)$. which is not zero

Hence there is only one soln to this system which is

$c_1 = c_2 = \cdots = c_n = 0$

$\therefore \varphi_1, \varphi_2 \ldots \varphi_n$ are linearly independent on I.

conversely,

Suppose $\varphi_1, \varphi_2 \ldots \varphi_n$ are linearly independent on I.

Suppose there is an x_0 in I $\ni$

$W(\varphi_1, \varphi_2 \ldots \varphi_n)(x_0) = 0.$

$\Rightarrow$ The system of n linear equations.

$$\left.\begin{aligned} &c_1\varphi_1(x_0) + c_2\varphi_2(x_0) + \cdots + c_n\varphi_n(x_0) = 0 \\ &c_1\varphi_1'(x_0) + c_2\varphi_2'(x_0) + \cdots + c_n\varphi_n'(x_0) = 0 \\ &\vdots \\ &c_1\varphi_1^{(n-1)}(x_0) + c_2\varphi_2^{(n-1)}(x_0) + \cdots + c_n\varphi_n^{(n-1)}(x_0) = 0 \end{aligned}\right\} \rightarrow (3)$$

has a soln $c_1, c_2 \ldots c_n$ where not all the constants $c_1, c_2 \ldots c_n$ are zero.

Let $c_1, c_2 \ldots c_n$ be such a soln and let us consider the function.

$\psi = c_1\varphi_1 + c_2\varphi_2 + \cdots + c_n\varphi_n$

Now $L(\psi) = 0$ and from (3) we see that,

$\psi(x_0) = 0, \quad \psi'(x_0) = 0 \quad \ldots \quad \psi^{(n-1)}(x_0) = 0$

$\therefore$ from uniqueness thm it follows that

$\psi(x) = 0 \quad \forall\ x$ in I

and hence,

$c_1\varphi_1(x) + c_2\varphi_2(x) + \cdots + c_n\varphi_n(x) = 0, \quad \forall\ x$ in I

But this contradicts the fact that $\varphi_1, \varphi_2 \ldots \varphi_n$ are L.I on I.

Thus our assumption that there is a point x_0 in I $\ni$ $W(\varphi_1, \varphi_2 \ldots \varphi_n)(x_0) = 0$ must be false.

$\therefore$ Hence $W(\varphi_1, \varphi_2 \ldots \varphi_n)(x) \neq 0, \quad \forall\ x$ in I.

Theorem: 7

Let $\varphi_1, \varphi_2 \ldots \varphi_n$ be n linearly independent solns of $L(y)=0$ on an interval I. If φ is any soln of $L(y)=0$ on I it can be represented in the form, $\varphi = c_1\varphi_1 + c_2\varphi_2 + \ldots + c_n\varphi_n$.

where $c_1, c_2 \ldots c_n$ are constants. Thus any set of 'n' linearly independent solns of $L(y)=0$ on I is a basis for the solns of $L(y)=0$ on I

Proof:

Let x_0 be a point in I and

Suppose $\varphi(x_0) = \alpha_1$, $\varphi'(x_0) = \alpha_2 \ldots \varphi^{(n-1)}(x_0) = \alpha_n$

we show that there exist unique constants $c_1, c_2 \ldots c_n$ $\ni$

$\psi = c_1\varphi_1 + c_2\varphi_2 + \ldots + c_n\varphi_n$ is a soln of $L(y)=0$ satisfying

$\psi(x_0) = \alpha_1$, $\psi'(x_0) = \alpha_2 \ldots \psi^{(n-1)}(x_0) = \alpha_n$.

By uniqueness Thm, we have,

$$\varphi = \psi \quad \text{(or)}$$

$$\varphi = c_1\varphi_1 + c_2\varphi_2 + \ldots + c_n\varphi_n$$

The initial conditions for ψ are equivalent to the following equations for $c_1, c_2 \ldots c_n$.

$$\left.\begin{aligned} &c_1\varphi_1(x) + c_2\varphi_2(x) + \ldots + c_n\varphi_n(x) = \alpha_1 \\ &c_1\varphi_1'(x) + c_2\varphi_2'(x) + \ldots + c_n\varphi_n'(x) = \alpha_2 \\ &\vdots \\ &c_1\varphi_1^{(n-1)}(x) + c_2\varphi_2^{(n-1)}(x) + \ldots + c_n\varphi_n^{(n-1)}(x) = \alpha_n \end{aligned}\right\} \rightarrow (1)$$

This is a set of linear eqn for $c_1, c_2 \ldots c_n$.

The determinant of the coefficient is $W(\varphi_1, \varphi_2 \ldots \varphi_n)(x_0)$ which is not zero.

Since $\varphi_1, \varphi_2 \ldots \varphi_n$ are L.I (by above theorem).

$\therefore$ There is unique soln $c_1, c_2 \ldots c_n$ of (1)

Hence the theorem.